INTEGRATION OF THE HAMILTON-JACOBI EQUATION BY THE METHOD OF SEPARATION OF VARIABLES

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The works of Aksenov, Grebenikov, Demin [1] and Kislik [2] on the analytic theory of the motion of artificial satellites of the Earth reveal the importance of a general investigation of canonical systems that can be integrated by the method of separation of variables.

Below is given a relatively simple method for the construction of the complete integral with separated variables of the Hamilton-Jacobi equation. The type of the equation can be arbitrary. Special attention is given however, to the case when the left-hand side of the equation is the sum of homogeneous polynomials of the first and zero degree with respect to the impulses, while the coefficients of these polynomials depend not only on the space coordinates but also on time. Under rather weak hypotheses, the considered method yields necessary and sufficient conditions for the integrability.* These conditions are imposed on the characteristic function of the problem. A method is given for selecting all those equations which can be integrated after a change of coordinates by means of a contact transformation. It is proved that if the Hamilton-Jacobi equation is of the same type as the equation of motion of a material point in an n-dimensional Euclidean space, then it is integrable only in ellipsoidal coordinates and its degenerations.

After Jacobi [3, p.6] had established, in 1843, by means of a rigorous mathematical analysis, that the principal Hamilton function

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* For the sake of brevity the words "by the method of separation of variables" will be omitted in the sequel.

can be found not just from two equations, as Hamilton had stated, but even from just one first order partial differential equation, there arose the problem of solving this equation. The method of separation of variables was found to be quite effective for certain types of equations. Jacobi showed in the work just mentioned that the Hamilton-Jacobi equation of the type

$$\tfrac{1}{2}(p_1^2 + \ldots + p_n^2) = h \qquad (p_i = \partial V / \partial x^i) \tag{0.1}$$

is integrable not only in rectangular coordinates $x^1, \ldots, x^n$, but also in ellipsoidal coordinates. The representation of the x^i in terms of ellipsoidal coordinates he called a "remarkable" substitution.

Soon after this, there arose the problem of integrating a Hamilton-Jacobi equation of a more general type*

$$\tfrac{1}{2}g^{ij}p_i p_j - U = h \qquad (p_i = \partial V / \partial q^i;\ i, j = 1, \ldots, n) \tag{0.2}$$

In 1847 Liouville found a case when the equation is integrable. He proved also that equation (0.1), with $n = 2$, can be integrated only in terms of elliptical coordinates and their degenerations (polar coordinates and similar ones).

In 1865, V.G. Imshenetski's candidate's thesis, and in 1869 his doctor's dissertation [4], were published in which the method of separation of variables was applied not only to the Hamilton-Jacobi equation but to a general first order partial differential equation. His idea is used in this paper.

In 1880 Morera [5] found two cases of integrability of equation (0.2) with $n = 2$. Independently of him, the same cases of integrability were found by Stäckel [6] in 1891. In this work Stäckel gave also a quite general case of integrability for equation (0.2) which was a generalization of Liouville's case. Stäckel [4] proved in 1893, that this case is the most general one for equation (0.2) which contains only squares of the impulses

$$\frac{1}{2}\sum_{i=1}^{n} g^{ii}p_i^2 - U = h \tag{0.3}$$

Stäckel's proof of the necessity and sufficiency for the integrability of equation (0.3) can be found in the monographs by Charlier [8] and by Lur'e [9].

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* In this paper the symbols employed in tensor analysis are used.

After the appearance of Stäckel's results there remained the problem of finding cases of integrability of equation (0.2) which contained products of impulses to different degrees, impulses to the first power, and the time explicitly in the characteristic function

$$H^* = p + {}^1/_2\, g^{ij} p_i p_j + h^i p_i - U = 0 \quad (p = \partial V / \partial t;\ i, j = 1, \ldots, n) \tag{0.4}$$

In 1904 Levi-Civita gave, in a letter to Stäckel [10], necessary and sufficient conditions for the integrability for equation (0.2). His conditions were expressed in terms of a system of partial differential equations for the characteristic function $H = H(q^1, \ldots, q^n, p_1, \ldots, p_n)$ of the problem

$$\frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial H}{\partial q^i} \frac{\partial H}{\partial q^j} - \frac{\partial^2 H}{\partial p_i \partial q^j} \frac{\partial H}{\partial p_j} \frac{\partial H}{\partial q^i} - \frac{\partial^2 H}{\partial q^i \partial p_j} \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q^j} + \frac{\partial^2 H}{\partial q^i \partial q^j} \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} = 0 \tag{0.5}$$
$$(i \neq j;\ i, j = 1, \ldots, n)$$

This system is valid also for equation (0.4) if one denotes its left-hand side by H and if t is treated as one of the space coordinates.

Levi-Civita substituted into this system the characteristic function of problem (0.2) and obtained in equations (0.5) a sum of homogeneous polynomials in p_i of degrees four, two and zero. Because of the arbitrariness of the constants of integration in the p_i, all the coefficients in these polynomials had to be zero. Equating to zero the coefficients in the fourth degree polynomial, one obtains a system of partial differential equations for g^{ij}. Equating to zero the coefficients of the terms of the second degree in p_i and of the free term, one obtains partial differential equations for g^{ij} and U. From this Levi-Civita deduced the theorem: if equation (0.2) is integrable then the equation is integrable with $U = 0$, i.e. in the absence of a force. An analogous argument can be made also for equation (0.4).

Levi-Civita separated the integration of the equations for g^{ij} into several cases. One of these cases, for arbitrary n, could be carried out completely. This case differs from Stäckel's case, but it has little use because for it U is necessarily zero. The consideration of the remaining cases is exceptionally involved; for this reason Levi-Civita confined himself to the case when $n = 2$, and he confirmed once more the results obtained by Morera [5] and Stäckel [6,7].

In 1908 Dall'Acqua gave explicitly all the partial derivatives for g^{ij}, which can be obtained from the equations of Levi-Civita, and investigated them for $n = 3$. He obtained all four types of integrable equations (for $n = 2$ there exist three types). In our work [12] it is proved, on the basis of the integrable types of Dall'Acqua that the existence of the remarkable "substitution of Jacobi" is not only sufficient but also a necessary condition that equation (0.1) may be integrable by the method of separation of variables when $n = 3$.

In 1911 Burgatti [11], on the basis of Levi-Civita's [10], and Dall'-Acqua's [11] studies, gave explicitly the expressions for the impulses in terms of arbitrary constants of integration α_j, h, which are the most general ones when $n = 2, 3$, and these expressions appear to be the most general ones for any n. He, however, was unable to prove that they actually are the most general ones for $n \geqslant 4$. The expressions of Dall'-Acqua [11] and of Burgatti [13] are not given because they are special cases of those found in Section 1.

It has not been established that the cases of Burgatti are the most general ones for any n. It is true that a number of cases of integrability of various generalizations of the Hamilton-Jacobi equation have been found. We regret that a complete listing of the works on this topic is not available; neither does there exist a survey article on this subject. We have mentioned briefly only the works known to us.

G.N. Duboshin, in his additions to the translation of Moulton's book [14], mentions that Moiseev has extended the case of integrability of Liouville to a characteristic function of the type $H = T_2 - T_0 - U$ ($T_0 \neq 0$). Demin [15] generalized Stäckel's case to apply to an equation that contains the impulses linearly.

But the possibilities here are limited. For example, Siegel [16] proved that in some quite general cases there does not even exist a contact transformation $(p, q) \to (\alpha, \beta)$ which can be expressed analytically (α, β are constants of integration).

1. We shall establish new cases of integrability of equation (0.4).

Theorem 1.1. Let an integer r be given $(0 \leqslant r \leqslant n)$, and let r^2 continuous functions $\vartheta^{ij}(q^i)$ $(i, j = 1, \ldots, r)$, and $(n - r)^2$ continuous functions $\varphi^{ij}(q^i)$ $(i, j = r + 1, \ldots, n)$ be given such that each of them depends only on one variable q^i and that the determinants $\Theta = \mathrm{Det} \, \| \vartheta^{ij} \|$ and $\Phi = \mathrm{Det} \, \| \varphi^{ij} \|$ are different from zero.

Furthermore, let these be given arbitrary continuous functions

$$\vartheta^{ij}(q^i),\ \sigma^{ikl}(q^i),\ \sigma^{ik}(q^i),\ \psi^i(q^i) \quad (i = r + 1, \ldots, n;\ j = 0, \ldots, r;\ k, l = 1, \ldots, r)$$

and arbitrary continuous functions of time

$$c^{kl}(t),\quad c^{ii}(t),\quad c^k(t),\quad \psi(t) \qquad (i = r + 1, \ldots, n;\ k, l = 1, \ldots, r)$$

Then the Hamilton-Jacobi equation (0.4) with the coefficients

$$g^{ij}=\frac{2}{\Theta^{2}}\sum_{k=1}^{r}\sum_{l=1}^{r}c^{kl}(t)\,\Theta^{ik}\Theta^{jl}+\qquad (i,\ j=1,\ \ldots,\ r)$$

$$+\frac{2}{\Theta^{2}\Phi}\sum_{m=r+1}^{n}\sum_{u=r+1}^{n}\sum_{k=1}^{r}\sum_{l=1}^{r}c^{uu}(t)\,\Phi^{mu}\Theta^{ik}\Theta^{jl}\left(\vartheta^{mk}\vartheta^{ml}-\sigma^{mkl}\right) \tag{1.1}$$

$$g^{ij}=-\frac{2}{\Theta\Phi}\sum_{u=r+1}^{n}\sum_{k=1}^{r}c^{uu}(t)\,\Phi^{ju}\Theta^{ik}\vartheta^{jk}\qquad \begin{pmatrix} i=1,\ \ldots,\ r\\ j=r+1,\ \ldots,\ n\end{pmatrix} \tag{1.2}$$

$$g^{ii}=\frac{2}{\Phi}\sum_{u=r+1}^{n}c^{uu}(t)\,\Phi^{iu}\quad (i=r+1,\ \ldots,\ n),\qquad g^{ij}=0\quad (i\neq j;\ i,j=r+1,\ldots,n) \tag{1.3}$$

$$h^{i}=-\frac{2}{\Theta^{2}}\sum_{k=1}^{r}\sum_{l=1}^{r}\sum_{j=1}^{r}c^{kl}(t)\,\Theta^{ik}\Theta^{jl}\vartheta^{j0}+\frac{1}{\Theta}\sum_{k=1}^{r}c^{k}(t)\,\Theta^{ik}-$$

$$-\frac{2}{\Theta^{2}\Phi}\sum_{m=r+1}^{n}\sum_{u=r+1}^{n}\sum_{k=1}^{r}\sum_{l=1}^{r}\sum_{j=1}^{r}c^{uu}(t)\,\Phi^{mu}\Theta^{ik}\Theta^{jl}\left(\vartheta^{mk}\vartheta^{ml}-\sigma^{mkl}\right)\vartheta^{j0}+ \tag{1.4}$$

$$+\frac{1}{\Theta\Phi}\sum_{m=r+1}^{n}\sum_{u=r+1}^{n}\sum_{k=1}^{r}c^{uu}(t)\,\Phi^{mu}\Theta^{ik}\left(2\vartheta^{mk}\vartheta^{m0}-\sigma^{mk}\right)\qquad (i=1,\ldots,\ r)$$

$$h^{i}=\frac{2}{\Theta\Phi}\sum_{u=r+1}^{n}\sum_{k=1}^{r}\sum_{l=1}^{r}c^{uu}(t)\,\Phi^{iu}\Theta^{kl}\vartheta^{il}\vartheta^{k0}-\frac{2}{\Phi}\sum_{u=r+1}^{n}c^{uu}(t)\,\Phi^{iu}\vartheta^{i0} \tag{1.5}$$

$$(i=r+1,\ \ldots,\ n)$$

$$U=-\frac{1}{\Theta^{2}}\sum_{k=1}^{r}\sum_{l=1}^{r}\sum_{m=1}^{r}\sum_{u=1}^{r}c^{kl}(t)\,\Theta^{mk}\Theta^{ul}\vartheta^{m0}\vartheta^{u0}+$$

$$+\frac{1}{\Theta}\sum_{k=1}^{r}\sum_{m=1}^{r}c^{k}(t)\,\Theta^{mk}\vartheta^{m0}+\psi(t)-$$

$$-\frac{1}{\Theta^{2}\Phi}\sum_{i=r+1}^{n}\sum_{j=r+1}^{n}\sum_{k=1}^{r}\sum_{l=1}^{r}\sum_{m=1}^{r}\sum_{u=1}^{r}c^{jj}(t)\,\Phi^{ij}\Theta^{mk}\Theta^{ul}\left(\vartheta^{ik}\vartheta^{il}-\sigma^{ikl}\right)\vartheta^{m0}\vartheta^{u0}+$$

$$+\frac{1}{\Theta\Phi}\sum_{i=r+1}^{n}\sum_{j=r+1}^{n}\sum_{k=1}^{r}\sum_{m=1}^{r}c^{jj}(t)\,\Phi^{ij}\Theta^{mk}\left(2\vartheta^{ik}\vartheta^{i0}-\sigma^{ik}\right)\vartheta^{m0}-$$

$$-\frac{1}{\Phi}\sum_{i=r+1}^{n}\sum_{j=r+1}^{n}c^{jj}(t)\,\Phi^{ij}\left[(\vartheta^{i0})^{2}-\psi^{i}\right] \tag{1.6}$$

where Φ^{ij} and Θ^{ij} are the cofactors of the ith row and jth column of the determinants Φ and Θ, is integrable and has a complete integral of the form

$$V = \alpha_{n+1} + \int_t p\,dt + \sum_{i=1}^{n} \int_{q^i} p_i dq^i \qquad (p = \partial V / \partial t,\ p_i = \partial V / \partial q^i) \quad (1.7)$$

where

$$p = -\sum_{k=1}^{r} \sum_{l=1}^{r} c^{kl}(t)\, \alpha_k \alpha_l - \sum_{u=r+1}^{n} c^{uu}(t)\, \alpha_u - \sum_{k=1}^{r} c^k(t)\, \alpha_k + \psi(t) \quad (1.8)$$

$$p_i = \vartheta^{i0}(q^i) + \sum_{k=1}^{r} \vartheta^{ik}(q^i)\, \alpha_k \qquad (i = 1, \ldots, r) \quad (1.9)$$

$$p_i = \vartheta^{i0}(q^i) + \sum_{k=1}^{r} \vartheta^{ik}(q^i)\, \alpha_k + $$
$$+ \left[\sum_{k=1}^{r} \sum_{l=1}^{r} \sigma^{ikl}(q^i)\, \alpha_k \alpha_l + \sum_{k=1}^{r} \sigma^{ik}(q^i)\, \alpha_k + \psi^i(q^i) + \sum_{u=r+1}^{n} \varphi^{iu}(q^i)\, \alpha_u \right]^{1/2} \quad (1.10)$$
$$(i = r+1, \ldots, n)$$

Proof. We shall make use of the results of Imshenetskii [4], and we shall show that the elimination of the arbitrary constants $\alpha_1, \ldots, \alpha_n$ from equations (1.8) to (1.10) leads to an equation of type (0.4), the coefficients of which are determined by formulas (1.1) to (1.6). Indeed, equations (1.9) permit us to express $\alpha_j (j = 1, \ldots, r)$ in terms of $p_i (i = 1, \ldots, r)$ and of the elements of the matrix $\|\vartheta^{ij}\|^{-1}$ which exists because of hypothesis the Det $\|\vartheta^{ij}\| = \Theta \neq 0$. But then equations (1.10), in which the $\alpha_i (i = 1, \ldots, r)$ have been substituted, make it possible to find the $\alpha_u (u = r + 1, \ldots, n)$ since by hypothesis Det $\|\varphi^{ij}\| = \Phi \neq 0$. The substitution of the expressions for $\alpha_i (i = 1, \ldots, n)$ into (1.8) will yield an equation of type (0.4), whereby (as can be shown by means of quite lengthy calculations) the still undetermined coefficients of this equation can be found with the aid of formulas (1.1) to (1.6). It remains only to verify that Det $\|\partial^2 V / \partial q^i \partial \alpha_j\| \neq 0$. But this determinant, for equations (1.9) and (1.10), is equal to $A\Theta\Phi$, where

$$A = 2^{-(n-r)} \prod_{u=r+1}^{n} \left(p_u - \vartheta^{u0} - \sum_{k=1}^{r} \vartheta^{uk} \alpha_k \right)^{-1} \neq 0$$

because infinitely large values of p_u and α_k are not considered, and because the ϑ^{uk} are continuous. Thus Det $\|\partial^2 V / \partial q^i \partial \alpha_j\| \neq 0$ since $\Theta \neq 0$ and $\Phi \neq 0$ by hypotheses.

Equation (1.7) makes it possible to find V by means of a simple quadrature because each impulse depends only on one coordinate. This proves the theorem.

This theorem establishes the integrability of the Hamilton-Jacobi equation with a characteristic function depending explicitly on time. This case has not been considered in the literature known to us. Therefore, a comparison with known cases can be made only for $\partial H/\partial t = 0$.

An analysis of [5-7, 10,11,13] shows that the above expressions for g^{ij}, U ($h^i = 0$) coincide with the most general ones for $n = 2, 3$, and with the expressions found by Stäckel [7] ($r = 0$), Burgatti [13] ($r \neq 0$), and Moiseev [14] for arbitrary n. If $h^i \neq 0$ expressions (1.1) to (1.6) include those found by Demin [15] (here $r = 0$).

One seeming generalization consists therein that the functions ϑ^{i0}, ϑ^{ij}, $-c^j$ and ψ are assumed to depend not only on one variable, but on all $t, q^1, \ldots, q^n$, and that they are subjected only to the integrability conditions

$$\partial p_i / \partial q^j = \partial p_j / \partial q^i, \qquad \partial p / \partial q^i = \partial p_i / \partial t$$

But this case can be reduced by means of a change of variables to one considered earlier (for more information about this substitution see Section 3).

The fact that the expressions found here cover all known cases is not an accident.

2. *Theorem 2.1.* In order that equation (0.4), which has a positive definite quadratic form $g^{ij}p_iq_j$ and continuous coefficients g^{ij}, h_i and U, may be integrated by the method of separation of variables, it is necessary and siffucient that the coefficients g^{ij}, h^i and U have the form indicated in Theorem 1.1.

Proof. We shall base our arguments again on ideas of Imshenetskii. It is known [4] that in a complete integral the arbitrary constants α are considered as depending on $t, q^1, \ldots, q^n$ (otherwise one obtains either a general or a particular integral). Therefore, if one solves the Hamilton-Jacobi equation (0.4) for each of the variables p, p_i and assumes that V is to be found in the form

$$V = V^0(t) + V^1(q^1) + \ldots + V^n(q^n)$$

then the left-hand side will contain functions of only that variable with respect to which the differentiation of V is performed, while the right-hand side will involve all variables. But all these variables, with the exception of the one on which the left-hand side depends, can be assumed to be equal to the initial values because of their independence. But then, if Det $\| \partial^2 V/\partial q^i \partial\alpha_j \| \neq 0$, the arbitrary constants which are in p and p_i can be chosen so that p and p_i will take on their initial values p^0 and p_i^0, whereby the set of initial values t_0, q_0^i, p^0 and p_i^0 must satisfy equation (0.4). Since the form $g^{ij}p_ip_j$ is positive definite in the p_i, $g^{ii} \neq 0$, and for each p_i there exists a quadratic equation from which it follows that

$$p_i = \vartheta_1^{\ i} + \vartheta_0^{\ i} \pm \sqrt{\theta_2^{\ i} + \theta_1^{\ i} + \theta_0^{\ i}} \tag{2.1}$$

Besides that

$$p = \theta_2 + \theta_1 + \theta_0 \tag{2.2}$$

All remaining arguments will be made for the case when the sign in front of the radical in (2.1) is plus. Analogous results hold when the sign is minus.

In formulas (2.1) and (2.2) the subscripts indicate the degree of the homogeneous polynomial in p_i^0 and p^0 which is denoted by the given letter. The initial value p^0 is given by the formula

$$p^0 = -\frac{1}{2} g_0^{\ ij} p_i^{\ 0} p_j^{\ 0} - h_0^{\ i} p_i^{\ 0} + U_0 \tag{2.3}$$

where the index 0 indicates that in g^{ij}, h^i and U we have set $t = t_0$ and $q^i = q_0^{\ i}$.

If one substitutes expressions (2.1) and (2.2) into equation (0.4) then one obtains an identity in t, q^i, t_0, $q_0^{\ i}$, $p_0^{\ i}$ if one puts p^0 from formula (2.3) into $\theta_2^{\ i} + \theta_1^{\ i} + \theta_0^{\ i}$, which we shall denote by $\varphi_2^{\ i} + \varphi_1^{\ i} + \varphi_0^{\ i}$ in what follows. The identity now takes the form

$$\theta_2 + \theta_1 + \theta_0 + \frac{1}{2} g^{ij} (\vartheta_1^{\ i} + \vartheta_0^{\ i} + \sqrt{\varphi_2^{\ i} + \varphi_1^{\ i} + \varphi_0^{\ i}})(\vartheta_1^{\ j} + \vartheta_0^{\ j} + \sqrt{\varphi_2^{\ j} + \varphi_1^{\ j} + \varphi_0^{\ j}}) + h^i (\vartheta_1^{\ i} + \vartheta_0^{\ i} + \sqrt{\varphi_2^{\ i} + \varphi_1^{\ i} + \varphi_0^{\ i}}) - U = 0 \tag{2.4}$$

and determines the form of the coefficients g^{ij}, h^i and U for which the variables in V are separated. The analysis of this identity depends mainly on whether the expressions $\varphi_2^{\ i} + \varphi_1^{\ i} + \varphi_0^{\ i}$ under the radical are perfect squares or not. In the general case one may assume that $\varphi_2^{\ i} + \varphi_1^{\ i} + \varphi_0^{\ i}$ is a perfect square when $i = 1, \ldots, r$. Then

$$\vartheta_1^{\ i} + \vartheta_0^{\ i} + \sqrt{\varphi_2^{\ i} + \varphi_1^{\ i} + \varphi_0^{\ i}} = \chi_1^{\ i} + \chi_0^{\ i} \qquad (i = 1, \ldots, r) \tag{2.5}$$

In view of this, identity (2.4) takes on the form

$$\theta_2 + \theta_1 + \theta_0 + \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} g^{ij} (\chi_1^{\ i} + \chi_0^{\ i})(\chi_1^{\ j} + \chi_0^{\ j}) +$$

$$+\sum_{i=1}^{r}\sum_{j=r+1}^{n} g^{ij}(\chi_1{}^i+\chi_0{}^i)(\vartheta_1{}^j+\vartheta_0{}^j)+\sum_{i=1}^{r}\sum_{j=r+1}^{n} g^{ij}(\chi_1{}^i+\chi_0{}^i)\sqrt{\varphi_2{}^j+\varphi_1{}^j+\varphi_0{}^j}+$$

$$+\frac{1}{2}\sum_{i=r+1}^{n}\sum_{j=r+1}^{n} g^{ij}(\vartheta_1{}^i+\vartheta_0{}^i)(\vartheta_1{}^j+\vartheta_0{}^j)+\frac{1}{2}\sum_{i=r+1}^{n} g^{ii}(\varphi_2{}^i+\varphi_1{}^i+\varphi_0{}^i)+$$

$$+\sum_{\substack{i,\,j=r+1\\ i>j}}^{n} g^{ij}(\vartheta_1{}^i+\vartheta_0{}^i)\sqrt{\varphi_2{}^j+\varphi_1{}^j+\varphi_0{}^j}+$$

$$+\sum_{\substack{i,j=r+1\\ i>j}}^{n} g^{ij}\sqrt{\varphi_2{}^i+\varphi_1{}^i+\varphi_0{}^i}\sqrt{\varphi_2{}^j+\varphi_1{}^j+\varphi_0{}^j}+\sum_{i=1}^{r} h^i(\chi_1{}^i+\chi_0{}^i)+$$

$$+\sum_{i=r+1}^{n} h^i(\vartheta_1{}^i+\vartheta_0{}^i)+\sum_{i=r+1}^{n} h^i\sqrt{\varphi_2{}^i+\varphi_1{}^i+\varphi_0{}^i}-U=0 \qquad (2.6)$$

Because of the arbitrariness of the initial values of the impulses, it follows that

$$g^{ij}=0 \qquad (i\neq j,\ i,j=r+1,\ldots,n) \qquad (2.7)$$

and, therefore, $\vartheta_1{}^i$ will depend only on $p_1{}^0, \ldots, p_r{}^0$, and not on $p_{r+1}^0, \ldots, p_n{}^0$. On the other hand, for the same reason $\varphi_2{}^i$ does not contain products $p_l{}^0 p_m{}^0 (l \neq m;\ l, m = r+1, \ldots, n)$. Hence, $\varphi_2{}^i + \varphi_1{}^i + \varphi_0{}^i$ is equal $(-\vartheta_1^i + \chi_1{}^i - \vartheta_0{}^i + \chi_0{}^i)^2$ only when $\chi_1{}^i$ also does not depend on $p_{r+1}^0, \ldots, p_n{}^0$. The introduction of the notation

$$\chi_1{}^i=\sum_{j=1}^{r}\vartheta^{ij}(q^i)\,p_j{}^0,\qquad \chi_0{}^i=\vartheta^{i0}(q^i)\qquad (i=1,\ldots,r) \qquad (2.8)$$

$$\vartheta_1{}^i=\sum_{j=1}^{r}\vartheta^{ij}(q^i)\,p_j{}^0,\qquad \vartheta_0{}^i=\vartheta^{i0}(q^i)\qquad (i=r+1,\ldots,n) \qquad (2.9)$$

$$\varphi_2{}^i=\sum_{k=1}^{r}\sum_{l=1}^{r}\sigma^{ikl}(q^i)\,p_k{}^0p_l{}^0+\sum_{k=1}^{r}\sum_{m=r+1}^{n}\sigma^{ikl}(q^i)\,p_k{}^0p_m{}^0+ \qquad (2.10)$$

$$+\sum_{m=r+1}^{n}\varphi^{im}(q^i)\,(p_m{}^0)^2 \qquad (i=r+1,\ldots,n)$$

$$\varphi_1^i=\sum_{k=1}^{r}\sigma^{ik}(q^i)\,p_k{}^0+\sum_{m=r+1}^{n}\sigma^{im}(q^i)\,p_m{}^0 \qquad (i=r+1,\ldots,n) \qquad (2.11)$$

$$\varphi_0{}^i=\psi^i(q^i) \qquad (i=r+1,\ldots,n) \qquad (2.12)$$

$$\theta_2 = -\sum_{k=1}^{r}\sum_{l=1}^{r} c^{kl}(t)\, p_k{}^0 p_l{}^0 - \sum_{k=1}^{r}\sum_{m=r+1}^{n} c^{km}(t)\, p_k{}^0 p_m{}^0 - \sum_{m=r+1}^{n} c^{mm}(t)\,(p_m{}^0)^2 \qquad (2.13)$$

$$\theta_1 = -\sum_{k=1}^{r} c^k(t)\, p_k{}^0 - \sum_{m=r+1}^{n} c^m(t)\, p_m{}^0, \qquad \theta_0 = \psi(t) \qquad (2.14)$$

and the equating to zero of the coefficients of the radicals and of the initial values of the impulses (because of their arbitrariness) yields systems of linear equations for the successive determination of the g^{ij}, h^i and U.

In order to find the g^{ii} $(i = r + 1, \ldots, n)$ it is sufficient to equate to zero the coefficients of $(p_i{}^0)^2$ $(i = r + 1, \ldots, n)$ in equation (2.6). Solving these equations for g^{ii}, we obtain a formula of type (1.3). If we equate to zero the coefficients of $p_i{}^0$, $p_j{}^0$ $(i = r + 1, \ldots, n;\ j = 1, \ldots, r)$ then we derive equations for the g^{ii} $(i = r + 1, \ldots, n)$ of the just described form. Hence $c^{kl}(t)$, $\sigma^{ikl}(q^i)$ $(k = 1, \ldots, r;\ l = r + 1, \ldots, n)$ must be linear combinations with constant coefficients, of $c^{jj}(t)$ and $\varphi^{ij}(q^i)$ $(i, j = r + 1, \ldots, n)$. The coefficients of g^{ij} $(i = 1, \ldots, r;\ j = r + 1, \ldots, n)$ are determined by the equations which express the vanishing of the coefficients of the radicals; this leads to expressions of type (1.2). Expressions of type (1.1) are obtained from the equations which insure the vanishing of the coefficients of $p_k{}^0 p_l{}^0$ $(k, l = 1, \ldots, r)$ by substituting into them the just determined coefficients g^{jj}, g^{ij} $(i = 1, \ldots, r;\ j = r + 1, \ldots, n)$. The vanishing of the coefficients of the first powers of $p_i{}^0$ $(i = r + 1, \ldots, n)$ yields expressions for h^i $(i = r + 1, \ldots, n)$ of type (1.5); while the vanishing of the coefficients of the first powers of $p_i{}^0$ $(i = 1, \ldots, r)$ yields expressions for h^i $(i = 1, \ldots, r)$ of type (1.4). Finally, the term which is free of $p_i{}^0$ gives an expression for the force function U of type (1.6).

In this manner expressions are found for all the coefficients which differ from expressions (1.1) to (1.6) only in the fact that their left-hand sides do not depend on the initial conditions t_0 and $q_0{}^i$, while their right-hand sides, formally speaking, do depend on them. By assigning to these intial terms some numerical values, we obtain formulas (1.1) to (1.6).

Since Det $\| \partial^2 V/\partial q^i \partial\alpha_j \| = A\Phi\Theta$ (Section 1) is not zero, $\Phi \neq 0$ and $\Theta \neq 0$. The theorem has thus been proved.

The method of finding the complete integral used in this proof leads to the same results as the equations of Levi-Civita, in the derivation of which it is also assumed that $\partial H/\partial p_i \neq 0$ [10]; our method is, however, considerably simpler. Naturally, it can be applied to Hamilton's equation of the most general type, as well as to an equation or a system of partial differential equations of any order. Besides that, this method does not require the solving of the canonical equations as was suggested by Imshenetskii [4].

From Theorems 1.1 and 2.1 it follows that the cases of Burgatti ($\partial H/\partial t = 0$) are the most general ones.

3. Expressions (1.1) to (1.6) can be considerably simplified even for the general case mentioned at the end of Section 1. This can be done by introducing new independent coordinates Q^i, of the time T and of the principal function W, the partial derivatives of which with respect to the old coordinates and time are given by the formulas

$$\frac{\partial Q^i}{\partial t} = -c^i, \qquad \frac{\partial Q^i}{\partial q^j} = \vartheta^{ij}, \qquad \frac{\partial Q^i}{\partial q^l} = 0, \quad \frac{\partial Q^k}{\partial t} = 0, \qquad \frac{\partial Q^k}{\partial q^j} = 0, \qquad \frac{\partial Q^k}{\partial q^l} = \delta_l^k$$

$$\frac{\partial W}{\partial t} = P = p + \psi, \quad \frac{\partial W}{\partial q^j} = P_j = p_j - \vartheta^{j0}, \quad \frac{\partial W}{\partial q^l} = P_l = p_l - \vartheta^{l0}, \quad T = t$$

$$(\delta_l^k = 1 \text{ if } k = l,\ 0 \text{ if } k \neq l;\ i, j = 1, \ldots, r;\ k, l = r+1, \ldots, n)$$

It is not difficult to verify that the conditions of integrability are here satisfied.

But then the corresponding Hamilton-Jacobi equation

$$P + {}^1/_2\, G^{ij} P_i P_j + H^i P_i - U^* = 0 \tag{3.1}$$

has the coefficients

$$G^{ij} = 2c^{ij}(T) - \frac{2}{\Phi} \sum_{k=r+1}^{n} \sum_{l=r+1}^{n} c^{ll}(T)\, \Phi^{kl} \sigma^{kij}(Q^k) \qquad (i, j = 1, \ldots, r)$$

$$G^{ii} = \frac{2}{\Phi} \sum_{k=r+1}^{n} c^{kk}(T)\, \Phi^{ik} \qquad (i = r+1, \ldots, n)$$

$$G^{ij} = 0 \qquad (i \neq j,\ i = 1, \ldots, n;\ j = r+1, \ldots, n)$$

$$H^i = -\frac{1}{\Phi} \sum_{k=r+1}^{n} \sum_{l=r+1}^{n} c^{ll}(T)\, \Phi^{kl} \sigma^{ki}(Q^k) \qquad (i = 1, \ldots, r)$$

$$H^i = 0 \qquad (i = r+1, \ldots, n)$$

$$U^* = \frac{1}{\Phi} \sum_{i=r+1}^{n} \sum_{j=r+1}^{n} c^{jj}(T)\, \Phi^{ij} \psi^i(Q^i)$$

The complete integral of equation (3.1) is given by formulas

$$W = \int_T P dT + \sum_{i=1}^{n} \int_{Q^i} P_i dQ^i, \quad P = -\sum_{k=1}^{r}\sum_{l=1}^{r} c^{kl}(T)\,\alpha_k\alpha_l - \sum_{j=r+1}^{n} c^{jj}(T)\,\alpha_j$$

$$P_i = \alpha_i \qquad (i = 1, \ldots, r) \tag{3.2}$$

$$P_j = \Big[\sum_{k=1}^{r}\sum_{l=1}^{r} \sigma^{jkl}(Q^j)\,\alpha_k\alpha_l + \sum_{k=1}^{r} \sigma^{jk}(Q^j)\,\alpha_k + \psi^j(Q^j) + \sum_{m=r+1}^{n} \varphi^{jm}(Q^j)\,\alpha_m\Big]^{1/2}$$

$$(j = r+1, \ldots, n)$$

Hence, such a change of variables simplifies considerably expressions (1.1) to (1.6). Formulas (3.2) show that the new impulses p_i $(i = 1, \ldots, r)$ retain their constant values; hence, coordinates Q^i $(i = 1, \ldots, r)$ are cyclical coordinates.

This analysis shows how important it is to find all those types of dependences of the principal function V on the arbitrary constants α of integration for which the elimination of these constants from $p = \partial V/\partial t$ and $p_i = \partial V/\partial q^i$ leads to an equation of type (0.4). The complete solution of this important problem, posed already by Burgatti [13], presents in the general case seemingly enormous difficulties, and is even impossible in the class of analytic functions [16].

4. We consider the following generalization of the results of Sections 1 and 2. As before, let V and p be given by formulas (1.7) and (1.8). For the p_i $(i = 1, \ldots, r)$ we have the formulas described at the end of Section 1, but the α_i $(i = r + 1, \ldots, n)$ are sums of homogeneous polynomials of degree not greater than two in p_j $(j = 1, \ldots, n)$ the coefficients of which depend on generalized coordinates and time. In this case the dependence of the p_j on the α_i $(i = 1, \ldots, n)$ will be, in general, irrational and even transcendental since, even when $n = 2$, one has to solve an algebraic equation of fourth degree in order to obtain this dependence.

Furthermore, the conditions of integrability [10] $\partial p_i/\partial q^j = \partial p_j/\partial q^i$ have to be fulfilled. These conditions are equivalent to equating the Poisson brackets equal to zero, i.e. $(\alpha_i, \alpha_j) = 0$ $(i, j = 1, \ldots, n)$. In these equations all coefficients of the various powers of p_i are zero because the constants of integration, which occur in the p_i, can take on arbitrary values. This yields partial differential equations in $g_k{}^{ij}$, $h_k{}^i$ and U_k if $\alpha_k = 1/2\ g_k{}^{ij} p_i p_j + h_k{}^i p_i - U_k$ $(k = r + 1, \ldots, n)$; even for $n = 2$ $(g_k{}^{12} \neq 0)$ these equations become very complicated; if, however, $g_k{}^{12} = 0$ we obtain the known Liouville-Stäckel case.

The possibility of another type of generalization is given by the solution of the Hamilton-Jacobi equation for the inertial motion of a material particle in a rotating system of coordinates

$$\tfrac{1}{2}(p_x^2 + p_y^2 + p_z^2) + n(yp_x - xp_y) = h$$

which cannot be integrated by the methods presented above even though the complete integral of the equation

$$p + \tfrac{1}{2}(p_x^2 + p_y^2 + p_z^2) + n(yp_x - xp_y) = 0$$

can be found by the method described at the end of Section 1. From it one can obtain the complete integral of the first equation; the dependence of V on α is transcendental and very complicated. Therefore, the "usual" substitution $p = -h$ can considerably complicate the integration of equation (0.4) even when $\partial H/\partial t = 0$.

The formulas of Section 1 show that the equations of the characteristics (the canonical equations) for integrable equations of type (0.4) have first integrals which are either quadratic or linear in the impulses. This fact makes it possible to solve effectively certain problems of stability for such systems by the method of combining first integrals (i.e. by the method of N.G. Chetaev). This method can be applied also to the system described in this section.

5. Again, let the Hamilton-Jacobi equation (0.4) be given with the functions g^{ij}, h^i and U of t and q^i. It is required to determine whether this equation can be integrated by the method of separation of variables. The easiest way to answer this is with the aid of a method which was used already by us in the proof of Theorem 2.1, because this method does not require the verification of the very complicated expressions for g^{ij}, h^i and U found in Section 1. This method can be reduced to the performance of the following operations:

a) We solve equation (0.4) for each of its partial derivatives p and p_i.

b) Each of the coordinates q^i (and time) except the one which is on the left-hand side, is equated to its initial value; this results in equating the impulses, which appear on the right-hand side, also to their initial values. Hereby, the initial values of the coordinates of time and of the impulses must, naturally, satisfy equation (0.4).

c) We integrate the expressions obtained with respect to that variable which occurs in the left-hand side, and add the result, constructing in this manner the principal function V.

d) We substitute the partial derivatives p and p_i into (0.4) and verify whether or not (0.4) becomes an identity $0 = 0$ if the initial values satisfy the condition given in subsection (b). In the first case equation (0.4) can be integrated by the method of separation of variables, in the second case it cannot be so integrated.

If equation (0.4) cannot be integrated by the above mentioned method then one has to try another procedure: one can check whether it can be integrated by the proposed method after a change of variables. For example, the two body problem can be integrated by this method in polar coordinates, but not in rectangular coordinates. There arises the general question: is it possible to give criteria on the basis of which one can determine whether or not there exist such transformations of variables $(q, t, p) \to (q', t', p')$ that the equation is integrable in the variables (q', t', p')?

Let us first analyze the case when $\partial H/\partial t = 0$, and when we assume that in equation (0.2) the space coordinates q only are transformed into the coordinates $q'(q \to q')$. In this case one may make use of a theorem of Levi-Civita [10] which states that if equation (0.2) is integrable then it is integrable if $U = 0$. According to results obtained in tensor analysis (see, for example, [17]), when $q \to q'$ then

$$R_{l'k'i'j'} = \frac{\partial q^l}{\partial q'^{l'}} \frac{\partial q^k}{\partial q'^{k'}} \frac{\partial q^i}{\partial q'^{i'}} \frac{\partial q^j}{\partial q'^{j'}} R_{lkij} \tag{5.1}$$

when R_{lkij} is the Riemann curvature tensor corresponding to the metric g^{ij}, while $R_{l'k'i'j'}$ is the Riemann curvature tensor for the metric $g'^{i'j'}$. Equations (5.1) solve the stated problem. Indeed, into their left-hand sides one can substitute expressions $g'^{i'j'}$ of Section 3, and into the right-hand sides expressions R_{lkij} in terms of the original coordinates which are known. Making use of the formulas for the transformation of the g^{ij}, one must eliminate from equations (5.1) all derivatives $\partial q^i/\partial q'^{i'}$, of which there are n^2, while the number of equations in (5.1) is at least n^3 (Section 5). The derived equations for the functions φ and σ with indices can be solved by generalizing the method of "fixing the extraneous variables" which was mentioned in Section 2. Substituting the expressions found for φ and σ into equations (5.1), and solving them for each of the derivatives $\partial q^i/\partial q'^{i'}$, one can obtain the formulas for the transformation $q \to q'$. One can obtain analogous results also in the more general case (Section 3) when there exist only square integrals which are in the involution since the mentioned theorem of Levi-Civita is valid here also.

For substitutions of the type $(q, t) \to (q', t')$, a generalization of the equations in (5.1) can be made by the methods of tensor analysis. For more general substitutions $(q, t, p) \to (q', t', p')$, the form of equation (0.4) in general is lost entirely, and the derivation of a generalization of (5.1) becomes very involved. One can obtain equations in partial derivatives with respect to the generating function of the transformation S and the characteristic function H by making use of the equation of Levi-Civita (0.5) and of the general theory of contact transformations [9,16]. Even for $n = 2$, the equations contain about ten thousand terms. It is much simpler to do this with the aid of the rules presented at the beginning of this section by applying them to the Hamilton-Jacobi equation of the most general type. It is quite possible that some of the simplifications may here be connected with the introduction of the special mathematical formalism. Even though the idea of the derivation of a generalization of equations (5.1) is relatively simple, the complicated details of the derivations will not be given here.

It is not difficult to see that the method just described can be used for solving the following important problem of finding all types of characteristic functions H for which the equation $p + H = 0$ can be integrated by the Hamilton-Jacobi method. Indeed, after the performance of the contact transformation $(q, t, p) \to (\alpha, \beta)$ the equation $(p + H)_{\alpha,\beta} = 0$, which is expressed in terms of α and β, must possess a complete integral $V = \alpha_1\beta_1 + \ldots + \alpha_n\beta_n$, i.e. the variables in V are separated in terms of the new space coordinates and time (either α or β). Since not every characteristic function H satisfies the corresponding equations (see the preceding paragraph), the Hamilton-Jacobi method can not be used for integrating every equation $p + H = 0$. This fact has been mentioned already earlier in [16], but no method was given there for identifying all integrable equations.

Since it is very difficult to solve the equations mentioned for S and H, the following procedure is of interest. Let $H = H_0 + H^{(1)}$, where the equation with H_0 is integrable* and introduces constants $\alpha^{(1)}$, $\beta^{(1)}$. Let us change the variables in $H^{(1)}$ to $\alpha^{(1)}$ and $\beta^{(1)}$, and let us separate that part H_1 which corresponds to the integrable case. This process can be continued. If $H = H_0 + H_1 + \ldots + H_N$, where N is finite, then the process will terminate at the Nth stage. If N is infinite then there arise two questions. The first one: is it possible to represent the function H in this form? The second one: may one apply the Hamilton-Jacobi method to such infinite expansions? It can happen that under reasonable restrictions on H or for a limited class of trajectories, or for a finite interval of time, the answers may be in the affirmative.

It must, however, be emphasized that some important problems become integrable after a general transformation $(q, t, p) \to (q', t', p')$ has been made; here we have in mind certain variants of problems in celestial mechanics [18-20].

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* See the first footnote of this paper.

Theorem 5.1. If equation (0.4) is integrable, so is its adjoint equation

$$\frac{\partial V'}{\partial t} - H\left(p, \frac{\partial V'}{\partial q}, t\right) = 0$$

Proof. Indeed, if we solve $p_i = p_i(q^i) = \partial V/\partial q^i$ for q^i we obtain $q^i = q^i(p_i) = \partial V'/\partial p_i$. This shows that the variables are separated in V'.

The form of equation (0.4) makes it possible to write out the canonical equations in which the independent variable can be any one of the coordinates q^i. It is sufficient for this to express equation (0.4) in the form $p_i + H_i = 0$, where H_i does not contain p_i. In the canonical system the independent variable may even be the variable q^{n+1} which does not occur in the left-hand side of equation (0.4). To show this, it is sufficient to add to the left-hand side of equation (0.4) the partial derivative $p_{n+1} = \partial V/\partial q^{n+1}$ assuming that V depends also on q^{n+1}.

6. The methods developed in the preceding sections can be applied to equation (0.1).

Theorem 6.1. If one transforms equation (0.1) from rectangular coordinates x^i to curvilinear coordinates q^i (that is, if one makes the substitution $q \to q'$) then the most general curvilinear coordinates q^i in which equation (0.1) is integrable by the method of separation of variables are the ellipsoidal coordinates or their degenerations (spherical or cylindrical coordinates).

This theorem states essentially that the Jacobi substitution [3] is not only sufficient but also a necessary condition for the integrability of equation (0.1).

Proof. For $n = 3$ this theorem was proved in [12]. For an arbitrary n it can be proved by the method of mathematical induction for the most interesting case $r = 0$. Since $\partial H/\partial t = 0$, we have the case of Stäckel. We make one additional hypothesis: the functions $\varphi^{ij}(q^i)$ are linearly independent. Since for the metric tensor corresponding to equation (0.1) all components of the Riemann curvature tensor are equal to zero, one has to solve the equations $R_{lkij} = 0$ and thus find the type of the functions φ^{ij} and the form of the substitution $q \to q'$. For orthogonal coordinates only the following components differ from zero, and they are independent of each other:

$$R_{ikij} = -\frac{1}{2}\frac{\partial^2 g_{ii}}{\partial q^k \partial q^j} + \frac{1}{4} g^{ii}\frac{\partial g_{ii}}{\partial q^j}\frac{\partial g_{ii}}{\partial q^k} + \frac{1}{4} g^{jj}\frac{\partial g_{ii}}{\partial q^j}\frac{\partial g_{jj}}{\partial q^k} + \frac{1}{4} g^{kk}\frac{\partial g_{kk}}{\partial q^j}\frac{\partial g_{ii}}{\partial q^k} \quad (i \neq j, j \neq k, k \neq i;\ i, j, k = 1, \ldots, n;\ g^{ii} = 1 : g_{ii}) \tag{6.1}$$

$$R_{ijij}=\frac{1}{2}\left(-\frac{\partial^2 g_{ii}}{\partial q^j\partial q^j}-\frac{\partial^2 g_{jj}}{\partial q^i\partial q^i}\right)+\frac{1}{4}g^{ii}\left[\left(\frac{\partial g_{ii}}{\partial q^j}\right)^2+\frac{\partial g_{ii}}{\partial q^i}\frac{\partial g_{jj}}{\partial q^i}\right]+$$
$$+\frac{1}{4}g^{jj}\left[\left(\frac{\partial g_{jj}}{\partial q^i}\right)^2+\frac{\partial g_{jj}}{\partial q^j}\frac{\partial g_{ii}}{\partial q^j}\right]-\frac{1}{4}\sum_{\substack{k=1\\ k\neq i,\,j}}^{n} g^{kk}\frac{\partial g_{ii}}{\partial q^k}\frac{\partial g_{jj}}{\partial q^k}\,(i\neq j;\ i,j=1,\ldots,n) \qquad (6.2)$$

These expressions are so complicated that even the substitution of g^{ii} for ellipsoidal coordinates q', ..., q^n yields zero only after quite long manipulations which are here considerably abbreviated. For ellipsoidal coordinates

$$g^{ii}=\frac{4f(q^i)}{(q^i-q^1)\ldots(q^i-q^n)},\qquad f(q^i)=(q^i-a_1^2)\ldots(q^i-a_n^2) \qquad (6.3)$$

Here of course, we have omitted $q^i - q^i$ from the denominator. Furthermore

$$\frac{\partial^2 g_{ii}}{\partial q^k\partial q^j}=\frac{g_{ii}}{(q^i-q^k)(q^i-q^j)},\quad g^{ii}\frac{\partial g_{ii}}{\partial q^j}\frac{\partial g_{ii}}{\partial q^k}=\frac{g_{ii}}{(q^i-q^j)(q^i-q^k)}$$
$$g^{jj}\frac{\partial q_{ii}}{\partial q^j}\frac{\partial g_{jj}}{\partial q^k}=\frac{g_{ii}}{(q^i-q^j)(q^j-q^k)},\qquad g^{kk}\frac{\partial g_{kk}}{\partial q^j}\frac{\partial g_{ii}}{\partial q^k}=\frac{-g_{ii}}{(q^j-q^k)(q^i-q^k)}$$

Thus, all the components $R_{lkij} = 0$. Next

$$\frac{\partial^2 g_{ii}}{\partial q^j\partial q^j}=\frac{\partial^2 g_{jj}}{\partial q^i\partial q^i}=0$$

We now establish the identity

$$g^{11}+\ldots+g^{nn}=-4(a_1^2+\ldots+a_n^2)+4(q^1+\ldots+q^n) \qquad (6.4)$$

If we make use of formula (6.3) and bring all fractions in the sum (6.4) to the common denominator

$$\prod_{k>l}^{n}(q^k-q^l)$$

then the numerator of the fraction becomes

$$4\sum_{m=0}^{n}\sum_{u=1}^{n}b_m(q^u)^m\prod_{\substack{k>l\\ k,l\neq u}}^{n}(q^k-q^l)=4\sum_{m=0}^{n}b_m\begin{vmatrix}(q^1)^m & (q^1)^{n-2}\ldots q^1 & 1\\ (q^2)^m & (q^2)^{n-2}\ldots q^2 & 1\\ \cdots & \cdots & \cdots\\ (q^n)^m & (q^n)^{n-2}\ldots q^n & 1\end{vmatrix}=$$
$$=4b_n\prod_{k>l}^{n}(q^k-q^l)(q^1+\ldots+q^n)+4b_{n-1}\prod_{k>l}^{n}(q^k-q^l)$$

where we have assumed that $f(q^i) = b_n(q^i)^n + b_{n-1}(q^i)^{n-1} + \dots + b_1 q^i + b_0$. This implies formula (6.4). Let us denote the last sum in (6.2) by S_{ij}. Since

$$\frac{\partial g_{ii}}{\partial q^k} = \frac{g_{ii}}{q^k - q^i}, \qquad \frac{\partial g_{jj}}{\partial q^k} = \frac{g_{jj}}{q^k - q^j}, \qquad \frac{\partial^2 g^{kk}}{\partial q^i \partial q^j} = \frac{g^{kk}}{(q^k - q^i)(q^k - q^j)}$$

then

$$S_{ij} = -\frac{1}{4} g_{ii} g_{jj} \frac{\partial^2}{\partial q^i \partial q^j} \sum_{\substack{k=1 \\ k \neq i,j}}^{n} g^{kk}$$

In view of (6.4)

$$S_{ij} = \frac{g_{ii} g_{jj}}{4} \frac{\partial^2 (g^{ii} + g^{jj})}{\partial q^i \partial q^j} = -\frac{g_{ii} + g_{jj}}{4(q^i - q^j)^2} + \frac{g^{ii} g_{jj}}{4(q^j - q^i)} \frac{\partial g_{ii}}{\partial q^i} + \frac{g_{ii} g^{jj}}{4(q^i - q^j)} \frac{\partial g_{jj}}{\partial q^j}$$

But the sum of the expressions in the two square brackets in (6.2) is equal to $-S_{ij}$. Hence, $R_{ijij} = 0$.

The converse argument goes as follows. Let the functions in the last row of the determinant Φ be arranged so that in some region of q^n

$$(\ln \varphi^{nk})' > (\ln \varphi^{nk+1})' \qquad (k = 2, \dots, n-1)$$

This is always possible since the functions φ^{ni} are linearly independent. But then it will be true that in some smaller region of q^n, $(\ln \varphi^{n1})'$ is either greater (Case 1) or less (Case 2) than $(\ln \varphi^{nn})'$. If one goes over to a new independent variable, $dq^{*n} = dq^n \cdot \varphi^{nn}$ in Case 1, or $dq^{*n} = dq^n : \varphi^{n2}$ in Case 2, then in the nth row there will stand the functions $\varphi^{*nk}(q^n) = \varphi^{nk} : \varphi^{nn} (\varphi^{*nn} = 1)$, or $\varphi^{*nk}(q^n) = \varphi^{nk} : \varphi^{n2} (\varphi^{*n2} = 1)$. Now one can substitute the determinant Φ with the transformed nth row into $R_{ikij} = 0$ and bring the obtained left-hand side to a common denominator. If i, j and $k \neq n$, then for this choice of the functions φ^{*ni}, not one of the products of the six functions φ^{*ni} will be a constant number. Because of the arbitrariness of q^n, the term which does not depend on φ^{*ni} must be equal to zero. This term is equal to the left-hand side of the equation $R^{(n-1)}_{ikij} = 0$ if Φ has $n-1$ rows and $n-1$ columns. But for the case of $n-1$ independent variables the theorem is true by hypothesis. The same arguments can be made if on substitutes functions of another variable q into the last row of the determinant Φ after one has rearranged the variables, and thus after the proper change of variables

$$\Phi = \begin{vmatrix} (\varphi^1)^{n-2}\varphi^{*1} & \dots & \varphi^1\varphi^{*1} & \varphi^{*1} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ (\varphi^n)^{n-2}\varphi^{*n} & \dots & \varphi^n\varphi^{*n} & \varphi^{*n} & 1 \end{vmatrix}$$

Let $i = 1$, $j = 2$ and $k = 3$. The coefficient of $(\Phi_{n-2,n-1,n})^6$, where $\Phi_{n-2,n-1,n}$ is the cofactor in Φ of the minor

$$\Phi^* = \begin{vmatrix} \varphi^1\varphi^{*1} & \varphi^{*1} & 1 \\ \varphi^2\varphi^{*2} & \varphi^{*2} & 1 \\ \varphi^3\varphi^{*3} & \varphi^{*3} & 1 \end{vmatrix}$$

in the equation $R_{ikij} = 0$ is equal to zero. For the determinant Φ^*, and its complements, we obtain an equation which has been considered in [12]. Hence

$$\Phi = \mathrm{Det} \,\| (\varphi^i)^{n-1}, \ldots, \varphi^i, \quad 1 \,\| = \prod_{k>l}^{n} (\varphi^k - \varphi^l)$$

if φ^i and φ^{*i} are linearly independent. After a proper change of variables, we deduce

$$g^{ii} = \prod_{k \neq i}^{n} (\varphi^i - \varphi^k) \tag{6.5}$$

We obtain the differential equation for the functions φ^i by substituting g^{ii} from (6.5) into $R_{ijij} = 0$. Hence, for $\varphi^k (k \neq i, j)$ we have

$$(d\varphi^k / dq^k)^2 = a_0^{(k)} (\varphi^k - a_1^{(k)}) \ldots (\varphi^k - a_n^{(k)}) \qquad (a_i^{(j)} = \mathrm{const})$$

Arguments analogous to those used in [12] yield $a_i^{(k)} = a_i$ $(i, k = 1, \ldots, n)$, and the following bounds for the functions:

$$\varphi^1 \geqslant a_1 \geqslant \varphi^2 \geqslant a_2 \geqslant \ldots \geqslant a_{n-1} \geqslant \varphi^n \geqslant a_n$$

Hence for the appropriate choices of a_i, $a_0^{(k)}$ and q^i

$$g_{ii} = \frac{1}{4} \prod_{k \neq i}^{n} [\wp (q^k) - \wp(q^i)] \tag{6.6}$$

However, if we select $\varphi^i = q^i$, we obtain (6.3). Therefore, the variables q^i are ellipsoidal coordinates in the space x^k.

If $r > 0$, and if there exists a linear relation among the functions $\varphi^{ij} (i = r + 1, \ldots, s)$, then the Q^i $(i = 1, \ldots, r)$ (Section 3) must be either angles which express an axial symmetry of the coordinate system, or they reduce to one of the Cartesian coordinates (compare cylindrical coordinates), while the Q^i $(i = r + 1, \ldots, s)$ are degenerate ellipsoidal coordinates (parabolic, spherical and so on). The corresponding proof by means of mathematical induction, however, becomes very involved due to the fact that one has to consider many subcases for large n. LeviCivita [10] has shown that when $r = n$ the coordinates Q^i can be only rectangular Cartesian coordinates. This completes the proof of the theorem.

The Liouville case of the integrability of equation (0.1) with $n > 2$ can hold only for the trivial case for rectangular Cartesian coordinates. This follows from the equation $R_{lkij} = 0$.

Thus the ellipsoidal coordinates are the most general ones for which equation (0.1) is integrable. But this means that one can find general conditions for h^i and U of Section 3. Since neither the force function of the n body problem ($n > 2$, $h^i = 0$), nor the expressions for h^i and U for the restricted circular and elliptical three body problem satisfy these conditions, these problems cannot be solved by the method of separation of space variables. This does not, however, imply that the results obtained here cannot be used in these problems. For example, one can seek a solution by the method of expanding the function H (Section 5). The results obtained are of practical value [19,20] if one takes for H_1 the characteristic function of the twice averaged restricted three body problem. Finally, one can find particular solutions of the indicated integrable problems.

No investigations have yet been made of the problem of determining the types of general contact transformations $(p, q) \to (p', q')$ which make equation (0.1) integrable in the variables p' and q' (see Section 5).

It is natural to pose also the problem of finding the most general coordinates for which the canonical equations with the characteristic equation (0.1) have integrals which are quadratic in p_i (see Section 3). If $n = 2$, and the coordinates are orthogonal, then they are elliptic coordinates (Section 3). But for $n > 2$, and for nonorthogonal coordinates with $n \geqslant 2$, the question is still open.

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